Casimir energy in a small volume multiply connected static hyperbolic pre-inflationary Universe

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A few years ago, Cornish, Spergel and Starkman (CSS), suggested that a multiply connected "small" Universe could allow for classical chaotic mixing as a pre-inflationary homogenization process. The smaller the volume, the more important the process. Also, a smaller Universe has a greater probability of being spontaneously created. Previously DeWitt, Hart and Isham (DHI) calculated the Casimir energy for static multiply connected flat space-times. Due to the interest in small volume hyperbolic Universes (e.g. CSS), we generalize the DHI calculation by making a a numerical investigation of the Casimir energy for a conformally coupled, massive scalar field in a static Universe, whose spatial sections are the Weeks manifold, the smallest Universe of negative curvature known. In spite of being a numerical calculation, our result is in fact exact. It is shown that there is spontaneous vacuum excitation of low multipolar components.

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I. INTRODUCTION

There has been an increase of interest in the topology of the Universe in recent years. As is well known, the Einstein equations (EQ) restrict the spatial homogeneous and isotropic sections to either R^3, S^3 or H^3, locally. Recent observational data indicate that the curvature of the Universe is small, without ruling out the negatively curved case. On the other hand, the EQ are insensitive to the global nontrivial topology, induced by a discrete group of isometries Γ acting freely and properly discontinuously in the covering space. This group acts according to a type of generalization of Poincaré's theorem - the tessellations of H^3 by hyperbolic polyhedra [1]. The polyhedra are pasted together, filling up the entire H^3, without leaving any empty space. The motions of the polyhedra, are performed by the discrete group of isometries Γ. For a review of topology in connection with cosmology, see [2] and references therein. Among the first applications of the topology considerations, was an attempt to explain multiple quasar images [3]. Constraints due to the homogeneity of the CMBR set a lower limit for the size of the fundamental cell today to $\sim 3000Mpc$ [4], [5] and [6]. The results apply only to compactifications of flat space. There are arguments, however, which allow for a compact hyperbolic manifold as the space section [7]. The effect of the topology induces the formation of circles in the sky, which, in principle could be measured in the CMBR [8].

A very attractive argument in favor of compact hyperbolic manifolds is connected with pre-inflationary homogenization through chaotic mixing, as was discussed by Cornish et al. [9]. They suggested that inflation occurred near the Planck era, after the chaotic homogenization, so that the two processes together are better suited to explain the large scale structure of the Universe observed today. The motivation for this work is based on this argument.

Due to the particular interest in the small volume hyperbolic Universe, we make a numerical calculation of the vacuum expectation value of the energy of a conformally coupled massive scalar field (e.g., inflaton) in a static space-time $R \times \mathcal{M}$, where the spatial section is the compact hyperbolic 3-manifold of the smallest volume known, $V = 0.942707...R^3_{\text{CURV}}$, called the Weeks manifold [10], $\mathcal{M}$, where R_{CURV} is the radius of curvature of the Universe. The quantum field theoretical effects were strongest in the era under consideration by Cornish et al. [9] than in any other. Previously, the Casimir energy for static multiply connected flat space-times, was obtained in DeWitt et al. [11]. We calculate a generalization of their result for a particular multiply connected hyperbolic space section. We use the point splitting technique which is very useful in determining the ultraviolet behavior in curved space, since it involves the evaluation of field quantities infinitesimally displaced. The obtained

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propagator is exact and possesses information about the global properties of the manifold in the sense that the infrared modes are taken into account, for Lie groups, as well as some homogeneous space-times, such as the static $R\times S^3$ and $R\times H^3$ [12]. When the space-time is multiply connected, the propagator is obtained as the usual sum over paths: all the nontrivial geodesics connecting the two points are taken into account, which is the technique used in [11], and the one used in this work.

We find a static hyperbolic solution for the EQ, whose spatial section is the Weeks manifold in section II. In section III, we write the expression for the vacuum expectation value of the energy momentum tensor. We obtain the numerical values of the Casimir energy in the multiply connected static space time of section II, in section IV. Our conclusions are presented in section V. (We use natural units, $G = c = \hbar = 1$, except in section II.)

II. THE STATIC UNIVERSE $R \times \mathcal{M}$

It is well known that hyperbolic geometry can be obtained via "Minkowski" space, together with an additional constraint

$$dl^2 = dx^2 + dy^2 + dz^2 - dw^2,$$
$$(x-x')^2 + (y-y')^2 + (z-z')^2 - (w-w')^2 = -R^2_{\text{CURV}}. \quad (1)$$

It can be easily seen that the isometry group is the proper, orthocronous Lorentz group, also called $SO(1,3)$, which is isomorphic to $PSL(2,C) = SL(2,C)/\{\pm I\}$ [13]. This space is homogeneous in the sense that every point in it can be reached from any other by the action of an element of the isometry group. Using the constraint in the line element of Eq. (1) we obtain

$$\begin{aligned} dl^2 &= dx^2 + dy^2 + dz^2 \\ &\quad - \frac{((x-x')\,dx + (y-y')\,dy + (z-z')\,dz)^2}{(x-x')^2 + (y-y')^2 + (z-z')^2 + R^2_{\text{CURV}}}, \\ dl^2 &= g(x,x')_{\mu\nu} dx^\mu dx^\nu, \end{aligned} \quad (2)$$

where we interchangeably write $\left(x^1, x^2, x^2\right) \longleftrightarrow (x,y,z)$. Both connections, ∇_x and $\nabla_{x'}$, compatible with the metric of Eq. (2), can be defined as

$$\nabla_\mu g(x,x')_{\alpha\beta} =$$
$$\frac{\partial}{\partial x^\mu} g(x,x')_{\alpha\beta} - \Gamma^\nu_{\alpha\mu} g(x,x')_{\nu\beta} - \Gamma^\nu_{\beta\mu} g(x,x')_{\alpha\nu} \equiv 0, \quad (3)$$
$$\nabla_{\mu'} g(x,x')_{\alpha\beta} =$$
$$\frac{\partial}{\partial x'^\mu} g(x,x')_{\alpha\beta} - \Gamma'^\nu_{\alpha\mu} g(x,x')_{\nu\beta} - \Gamma'^\nu_{\beta\mu} g(x,x')_{\alpha\nu} \equiv 0, \quad (4)$$

where Γ' means that all derivatives are taken with respect to x'. Spherical coordinates and the substitution

$$r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \quad (5)$$

in Eq. (2) yield the popular Robertson-Walker line element, written in the Lobatchevsky form

$$ds^2 = -dt^2 + R^2_{\text{CURV}} \left[d\chi^2 + \sinh^2\chi \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right], \quad (6)$$
$$\sinh^2\chi = \frac{r^2}{R^2_{\text{CURV}}} = \frac{(x-x')^2 + (y-y')^2 + (z-z')^2}{R^2_{\text{CURV}}}.$$

As is well known, the EQ for the homogeneous and isotropic space sections in Eq. (6) with $R_{\text{CURV}} = R_{\text{CURV}}(t)$, reduces to the Friedmann equations

$$\left(\frac{\dot R_{\text{CURV}}}{R_{\text{CURV}}}\right)^2 - \frac{1}{R^2_{\text{CURV}}} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3},$$
$$2\left(\frac{\ddot R_{\text{CURV}}}{R_{\text{CURV}}}\right) + \left(\frac{\dot R_{\text{CURV}}}{R_{\text{CURV}}}\right)^2 - \frac{1}{R^2_{\text{CURV}}} = -8\pi G p + \Lambda,$$

where the right hand side is the classical energy momentum source for the geometry $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}$, and a cosmological constant $\Lambda g^{\mu\nu}$.

During the stages just after the Planck era, it is likely that the Universe was radiation-dominated, so that we have the equation of state $\rho/3 = p$. Thus the energy density scaled as $\rho = \rho_0 (R_{\text{CURV}0}/R_{\text{CURV}}(t))^4$. We define $8\pi G \rho_0 R^4_{\text{CURV}}/3 = C$. By imposing that $\dot R_{\text{CURV}} = \ddot R_{\text{CURV}} = 0$ for a static Universe, we obtain from the above

$$\frac{C}{R^2_{\text{CURV}}} + 1 + \frac{\Lambda}{3} R^2_{\text{CURV}} = 0,$$
$$-C + \frac{\Lambda}{3} R^4_{\text{CURV}} = 0,$$

which have a solution

$$R_{\text{CURV}} = \sqrt{\frac{3}{2|\Lambda|}},$$
$$\rho = \frac{\Lambda}{8\pi G},$$
$$ds^2 = -dt^2 + R^2_{\text{CURV}} \left[d\chi^2 + \sinh^2\chi \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right], \quad (7)$$

where the cosmological constant is negative.

According to quantum cosmology, a smaller Universe has a greater probability of being spontaneously created. Also, the chaotic mixing becomes more significant in this case [9]. The smallest known compact 3-manifold $\mathcal{M}$, with volume $V = 0.942707...R^3_{\text{CURV}}$ was discovered by Weeks [10]. It is a multiply connected manifold with universal covering H^3, $\mathcal{M} = H^3/\Gamma$, with group $\Gamma \subset SO(1,3)$, a discrete finite subgroup with no fixed point. Group Γ is isomorphic to $\pi_1(\mathcal{M})$, the first homotopy group, also called the fundamental group of $\mathcal{M}$. $\pi_1(\mathcal{M})$ is the group of nontrivial loops composed of the maps of the manifold to the sphere $\mathcal{M} \to S^1$ [15]. For this smallest volume manifold, the 18 $SO(1,3)$ matrices g_i, which generate $\Gamma(0.942707...R^3_{\text{CURV}})$, were obtained with the computer program SnapPea [14]. The fundamental

domain is shown in FIG. 1. We note that the isometries preserve the form of the spatial part of the metric in Eq. (6), so that the Friedmann equations remain unaltered since they depend only on time t, and the solution of EQ is the same as in Eq. (7).

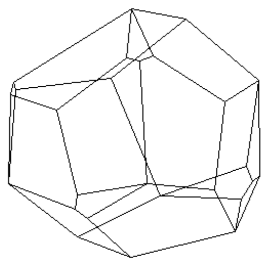

FIG. 1. Fundamental region with 18 faces in Klein coordinates, for the Weeks' manifold, the smallest volume manifold known.

III. THE VACUUM EXPECTATION ENERGY IN THE COVERING SPACE

We now wish to evaluate the vacuum expectation value for the energy density for the case of a Universe consisting of a classical radiation fluid, a cosmological constant, and a non interacting quantum scalar field. The solution of EQ is given in Eq. (7), where the spatial part is written as Eq. (2) and the topology is $R \times H^3/\Gamma(0.942797...R^3_{\text{CURV}})$. We use the point splitting method in the Universal covering space $R \times H^3$, for which the propagator is exact. The point splitting method was constructed to obtain the renormalized (finite) expectation values of the quantum mechanical operators. It is based on the Schwinger formalism [16] and was developed in the context of curved space by DeWitt [18]. Further details are contained in the articles of Christensen [19], [20]. For a review, see [21].

Metric variations in the scalar action S with conformal coupling $\xi = 1/6$,

$$S = -\frac{1}{2}\int \sqrt{g}(\phi_{,\rho}\phi^{,\rho} + \xi R\phi^2 + m^2\phi^2)d^4x,$$

give the classical energy momentum tensor

$$\begin{aligned} T_{\mu\nu} = {} & \frac{2}{3}\phi_{,\mu}\phi_{\nu} - \frac{1}{6}\phi_{\rho}\phi^{,\rho}g_{\mu\nu} - \frac{1}{3}\phi\phi_{;\mu\nu} \\ & + \frac{1}{3}g_{\mu\nu}\phi\Box\phi + \frac{1}{6}\phi^2 G_{\mu\nu} - \frac{1}{2}m^2 g_{\mu\nu}, \end{aligned} \tag{8}$$

where $G_{\mu\nu}$ is the Einstein tensor. As expected for massless fields, where $m = 0$, it can be verified that the trace of the above tensor is identically zero. Variations with respect to ϕ result in the curved space generalization of the Klein-Gordon equation,

$$\Box\phi - \frac{R}{6}\phi - m^2\phi = 0. \tag{9}$$

The renormalized energy momentum tensor involves field products at the same space-time point. Thus the idea is to calculate the products at separated points, x and x', taking the limit at the end $x \to x'$:

$$\langle 0|T_{\mu\nu}|0\rangle \sim \lim_{x\to x'} \nabla_\mu \nabla_{\nu'} \frac{1}{2}\langle 0|(\phi(x)\phi(x') + \phi(x')\phi(x))|0\rangle, \tag{10}$$

where the covariant derivatives are defined in Eqs. (3) and (4).

We introduce the causal Green function

$$G(x, x') = i\langle 0|T\phi(x)\phi(x')|0\rangle,$$

where T is the time ordering operator. Taking the real and imaginary parts of the Feynman propagator,

$$G(x, x') = G_s(x, x') + \frac{i}{2}G^{(1)}(x, x'), \tag{11}$$

results in the Hadamard function

$$G^{(1)}(x, x') = \langle 0|\{\phi(x), \phi(x')\}|0\rangle,$$

which is the expectation value of the anti-commutator $\{\phi(x), \phi(x')\} = \phi(x)\phi(x') + \phi(x')\phi(x)$, which appears in Eq. (10). By the above reasoning, taking into account Eq. (9) and the field products in the classical equation (8), the vacuum expectation value is obtained

$$\begin{aligned} & \langle 0|T_{\mu\nu}|0\rangle = \\ & \lim_{x\to x'} \left[\frac{1}{6}\left(\nabla_\mu\nabla_{\nu'} + \nabla_{\mu'}\nabla_\nu\right) - \frac{1}{12}g(x,x)_{\mu\nu}\nabla_\rho\nabla^{\rho'} \right. \\ & -\frac{1}{12}\left(\nabla_\mu\nabla_\nu + \nabla_{\mu'}\nabla_{\nu'}\right) + \frac{1}{48}g(x)_{\mu\nu}\left(\Box + \Box'\right) \\ & \left. +\frac{1}{12}\left(R(x)_{\mu\nu} - \frac{1}{4}R(x)g(x)_{\mu\nu}\right) - \frac{1}{8}m^2 g(x)_{\mu\nu}\right] \\ & G^{(1)}(x, x'), \end{aligned} \tag{12}$$

$$\langle 0|T_{\mu\nu}|0\rangle = \lim_{x\to x'}\left[T(x, x')_{\mu\nu}\right], \tag{13}$$

where the metric used to abtain the curvature tensor $R(x)_{\mu\nu}$, $g(x)_{\mu\nu} = g(x, x' = 0)_{\mu\nu}$, is given in (2), and the covariant derivatives are given in (3) and (4).

IV. THE FEYNMAN PROPAGATOR AND THE CASIMIR ENERGY IN $R \times \mathcal{M}$

The Feynman propagator involves a summation over all possible classical paths. When the space is multiply connected, it is necessary to sum over all nontrivial geodesics connecting two points. Green functions are solutions of Eq. (9) with the boundary condition

$$F(x,x')G(x,x') = \delta(x-x'),$$
$$F(x,x') = F(x)/\sqrt{-g}\delta(x-x'),$$

where $F(x) = \Box - R/6 - m^2$.

We introduced an auxiliary evolution parameter s and a complete orthonormal set of states $|x\rangle$, such that

$$G(x,x') = \langle x|\hat{G}|x'\rangle,$$
$$F(x,x') = \langle x|\hat{F}|x'\rangle,$$
$$\hat{F}\hat{G} = \hat{I}.$$

This last equation implies that $\hat{G} = (\hat{F} - i0)^{-1}$, so that the causal Green function becomes

$$G(x,x') = i\int_0^\infty ds\langle x|\exp(-is\hat{F})|x'\rangle \tag{14}$$

and the matrix element $\langle x|\exp(-is\hat{F})|x'\rangle = \langle x(s)|x'(0)\rangle$ satisfies a Schrödinger type equation,

$$i\frac{\partial}{\partial s}\langle x(s)|x'(0)\rangle = \left(\Box - R/6 - m^2\right)\langle x(s)|x'(0)\rangle.$$

Dowker and Critchley [17] obtained the Green function for the static homogeneous space-time with spherical space sections (S^3), using the above technique. Using a similar procedure, the result for a static hyperbolic space section is obtained. Assuming that $\langle x(s)|x'(0)\rangle$ depends only on the geodesic distance χ, given in Eq. (6), the above equation is easily solved. By substituting the solution $\langle x(s)|x'(0)\rangle$ for the integrand in Eq. (14),

$$G(x,x') = -\frac{m}{8\pi}\frac{\chi}{\sinh\chi}\frac{H_1^{(2)}\left(m\sqrt{(t-t')^2 - R_{\text{CURV}}^2\chi^2}\right)}{\sqrt{(t-t')^2 - R_{\text{CURV}}^2\chi^2}}, \tag{15}$$

where $H_1^{(2)}$ is the Hankel function of the second kind of order 1, the causal Green function is obtained.

The Klein-Gordon equation remains unchanged under isometries,

$$\pounds_\xi\left[\left(\Box - \frac{R}{6} - m^2\right)\phi\right] = \left(\Box - \frac{R}{6} - m^2\right)\pounds_\xi\phi,$$

where $\pounds_\xi$ is the Lie derivative with respect to ξ, is the Killing vector that generates the isometry, so that summations in the Green functions over the discrete elements of the group $\Gamma(0.942707...R_{\text{CURV}}^3)$ remains well defined. For comparison, we have from Eq. (6) in [17],

$$G(x,x',\kappa^2) = -\frac{\kappa^2}{8\pi a\sin(s/a)}$$
$$\sum_{n=-\infty}^{\infty}(s+2\pi na)\frac{H_1^{(2)}\left(\kappa\sqrt{(t-t')^2 - s^2}\right)}{\kappa\sqrt{(t-t')^2-s^2}}.$$

The infinite summation appears because the space sections S^3 are compact and, therefore, there are an infinite number of geodesics connecting two points. In fact, by removing the summation, leaving only the direct path and making the substitutions $s/a \to i\chi$, $a \to iR_{\text{CURV}}$, each formula may be derived from the other. The Hadamard function can be obtained from Eqs. (15) and (11),

$$G^{(1)}(x,x') = \frac{m}{2\pi^2}\frac{\chi}{\sinh\chi}\frac{K_1\left(m\sqrt{-(t-t')^2 + R_{\text{CURV}}^2\chi^2}\right)}{\sqrt{-(t-t')^2 + R_{\text{CURV}}^2\chi^2}}, \tag{16}$$

where K_1 is the modified Bessel function of the second kind.

Substituting Eq. (16) and the covariant derivatives (3) and(4) in Eq. (12), we obtain $T(x,x')_{\mu\nu}$ in Eq. (13). In $\mathcal{M} = R \times H^3/\Gamma(0.942707...R_{\text{CURV}}^3)$, the summation over the infinite geodesics connecting the two points x and x' is performed by the action of the generators g_i of the group $\Gamma(0.942707...R_{\text{CURV}}^3)$ and their products on the points x', (for example, in (13)):

$$\left\langle 0\left|T(x,0.942707...R_{\text{CURV}}^3)_{\mu\nu}\right|0\right\rangle = \lim_{x\to x'}\sum_i T\left(x,\Gamma_i x'\right)_{\mu\nu}. \tag{17}$$

We evaluated Eq. (17) numerically for the compact space-time $\mathcal{M}$. In the summation, the direct path gives a divergent contribution. It can be shown that avoiding the direct path is equivalent to a renormalization of the cosmological constant [21]. We shall use the same type of renormalization. In Eq. (17), we summed over the generators and their products up to three factors (see below) and assured that no transformed point in the covering space $\Gamma_i x'$ is summed more than once. We also checked that relation (1) was verified for each transformed point, $\Gamma_i x'$.

We obtained the result shown in FIG. 2, for a scalar field with mass $m = 0.5$, $R_{\text{CURV}} = 10$. The values of the vacuum energy, E, in FIG. 2, seen by an observer with a four velocity $u^\mu = (1,0,0,0)$ of

$$E = \left\langle 0\left|T(x,0.942707...R_{\text{CURV}}^3)_{\mu\nu}\right|0\right\rangle u^\mu u^\nu$$

at each point x on the surface of a sphere inside the fundamental polyhedron in FIG. 1. The radius of the sphere is chosen to be $r = 0.6$, where r is given in Eq. (5). θ and ϕ correspond to the latitude and longitude, so that the lines $\theta = 0$ and $\theta = \pi$ correspond to the south and north poles, respectively. It is clear from FIG. 2 that there are spontaneous vacuum excitations of low multipolar components.

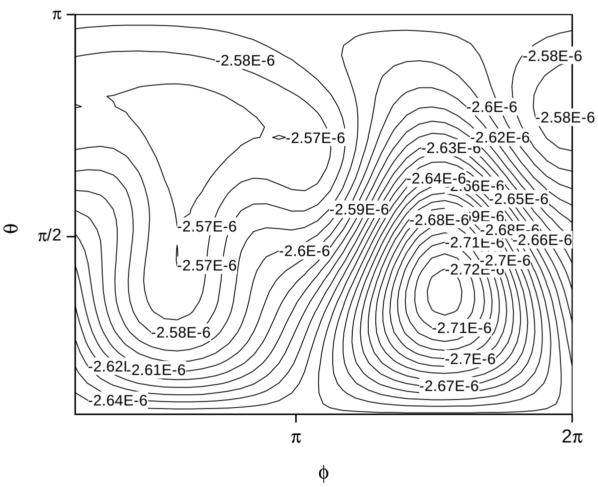

FIG. 2. Renormalized value of the vacuum 00 component of a conformally coupled massive scalar field ϕ ($E = T_{\mu\nu}u^{\mu}u^{\nu} = T_{00}$) with a mass $m = 0.5$ in the static $R \times \mathcal{M}$ space-time with $R_{\rm CURV} = 10$. The T_{00} is on a sphere of radius $r = 0.6$, inside the fundamental polyhedron, FIG. 1, where r is given in Eq. (5).

The infinite summation in Eq. (17) occurs because the space-time $R \times \mathcal{M}$ is static, so that there has been enough time for the quantum interaction of the scalar field with the geometry to travel through space. Since the Universe is expanding, we know that the summation ceases to be strictly physically valid. The presence of the mass term, however, naturally introduces a cutoff.

In Eq. (17), we summed over the generators and their products. We found that products of four generators or more, contributed less than 10^{-14} to the values in FIG. 2 (i.e. $\sim 10^{-8}$ of the plotted values of T_{00}).

V. CONCLUSIONS

We explicitly evaluated the distribution of the vacuum (Casimir) energy of a conformally coupled massive scalar field for a specific interesting situation: the smallest static Universe of negative curvature known, whose spatial sections are the Weeks manifold. As a specific example, we chose $m = 0.5$ for the mass of the scalar field, $R_{\rm CURV} = 10$ for the radius of curvature of the Universe and values of T_{00} on a sphere of radius $r = 0.5$ inside the fundamental polyhedron FIG. 1, are shown in FIG. 2. It can be seen in FIG. 2 that there is a maximum of $T_{00} \sim 2.6 \times 10^{-6}$ at $\theta \sim 3\pi/4$, $\phi \sim \pi/2$ and a minimum of $T_{00} \sim 2.7 \times 10^{-6}$ at $\theta \sim \pi/4$, $\phi \sim 3\pi/2$. It is to be noted that this type of topological effect depends on the mass: for smaller values of m the maximum values are bigger and the minimum values are smaller.

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